MIT OCW 8.03 Vibrations and Waves - Solutions to Problems Sets

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Problem Set 1

Solution 1:

(a) Intuitively it makes sense that the oscillations would not be dependent on gravity. This is because the oscillations would be the same as that on a vertical spring except this time it would be about a different equilibrium positions. Showing this mathematically is a slightly harder task.

First, we find the equilibrium position. We have our coordinate position to be the rest point at the center of mass of the block with the y-axis going upwards and the x-axis going horizontally. Let the extension in spring in equilibrium be y_0. There are two forces involved, the force from the spring and the force from gravity. This means that from a force analysis, we have that

$$ky_0 - mg = 0 \implies y_0 = \frac{mg}{k}.$$

Now displace the mass by y. From Newton's Law, we find that

$$\vec{F} = m\vec{a} = m\frac{d^2y}{dt^2} = m\ddot{y} = (mg - ky)\hat{y}.$$

Rewriting, we find that

$$m\ddot{y} = -k\left(y + \frac{mg}{k}\right)\hat{y} = -k(y - y_0).$$

We can define $x \equiv y - y_0$ which gives us the same equation as in the lecture with $\omega = \sqrt{\frac{k}{m}}$. Gravity affects only the equilibrium position not the angular frequency.

Solution 2:

(a) To take derivatives of the function

$$X(t) = Ae^{-i(\omega t - \pi/2)}$$

we must make use of Euler's identity which states

$$e^{ix} = \cos x + i\sin x.$$

This means that

$$X(t) = A\left[\cos\left(-\left(\omega t - \frac{\pi}{2}\right)\right) + i\sin\left(-\left(\omega t - \frac{\pi}{2}\right)\right)\right].$$

We make use that $\cos(x) = \cos(-x)$ and $\sin(-x) = -\sin(x)$ to simplify our expression to be

$$X(t) = A\left[\cos\left(\omega t - \frac{\pi}{2}\right) - i\sin\left(\omega t - \frac{\pi}{2}\right)\right] \implies X(t) = \boxed{A(\sin\omega t + i\cos\omega t)}.$$

Therefore, from taking the first derivative, we have that

$$\dot{X}(t) = -A\omega(i \sin \omega t - \cos \omega t).$$

The real component of this will be given by

$$\dot{X}(t) = \boxed{A\omega \cos \omega t}.$$

The second derivative of this function is then given as

$$\ddot{X}(t) = -A\omega^2(\sin \omega t + i \cos \omega t).$$

The real component of this will be given by

$$\ddot{X}(t) = \boxed{-A\omega^2 \sin \omega t}.$$

(b) We are given that

$$\dot{X}(t) = \boxed{Bi\omega\tau e^{-i(\omega t + \pi/6)}}$$

which we can rewrite as

$$\dot{X}(t) = \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right) Bi\omega\tau e^{-i\omega t}.$$

Integrating this gives us

$$X(t) = \int \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right) Bi\omega\tau e^{-i\omega t} \mathrm{dt} = -\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right) B\tau e^{-i\omega t}.$$

Using Euler's identity, we find that

$$X(t) = -\left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right) B\tau(\cos \omega t - i \sin \omega t)$$

the real component of $X(t)$ will be

$$X(t) = \boxed{-\frac{\sqrt{3}}{2} B\tau \cos \omega t}.$$

Differentiating $\dot{X}(t)$ with respect to t gives us

$$\ddot{X}(t) = \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right) B\omega^2\tau e^{-i\omega t} = \left(\frac{\sqrt{3}}{2} - \frac{i}{2}\right) B\omega^2\tau(\cos \omega t - i \sin \omega t).$$

Taking the real component tells us that

$$\ddot{X}(t) = \boxed{\frac{\sqrt{3}}{2} B\omega^2\tau \cos \omega t}.$$

Solution 3:

(a) On displacing by a distance x from the equilibrium position, both springs will have a spring force directed towards the equilibrium position. This means that we have

$$F_{\text{net}} = -(2Kx + Kx)\hat{x} = -3Kx\hat{x}.$$

Now, we write Newton's second law, and get rid of $\hat{x}$ as all the forces are in the x-direction to yield

$$m\ddot{x} = -3Kx \implies \omega = \boxed{\sqrt{\frac{3K}{M}}}.$$

(b) At mean position, $v = A\omega$. Therefore

$$A = \frac{v}{\omega} = \boxed{v\sqrt{\frac{M}{3K}}}.$$

(c) We know

$$x = A\sin(\omega t + \phi).$$

From initial conditions we know that $\omega = \sqrt{\frac{3K}{M}}$, $A = v\sqrt{\frac{M}{3K}}$, and $\phi = 0$. Therefore

$$x = \boxed{v\sqrt{\frac{M}{3K}}\sin\left(\sqrt{\frac{3K}{M}}t\right)}.$$

(d) From the lecture, we see that we can write the position of the center of mass as the complex function

$$z(t) = \text{Re}\left[Ae^{i(\omega t+\phi)}\right].$$

From initial conditions we know that $\omega = \sqrt{\frac{3K}{M}}$, $A = v\sqrt{\frac{M}{3K}}$, and $\phi = 0$ and therefore we have,

$$z(t) = \boxed{\text{Re}\left[v\sqrt{\frac{M}{3K}}e^{i\sqrt{\frac{3K}{M}}t}\right]}.$$

Solution 4:

(a) If we are given a potential $V(x)$, then the stable equilibrium positions of $V(x)$ will be found at the zeros of $V'(x)$. We find that the derivative of the potential is given by

$$V'(x) = 0 = 4x^3 + 12ax^2 - 16a^2x.$$

Factoring out $4x$ from both sides of the equation gives us

$$x^2 + 3ax - 4a^2 = 0.$$

Factoring this equation gives us

$$-(a-x)(4a+x) = 0 \implies x = a, 4a.$$

(b) As seen in the lecture, we have that

$$F(x) \approx -V^{''}(x)x \implies \omega = \sqrt{\frac{V''(x)}{m}}.$$

By taking the second derivative, we have that

$$V''(x) = 2x + 3a.$$

We can evaluate $V''(x)$ at both of our equilibrium points $x = a, 4a$. At $x = a$,

$$V''(a) = 5a \implies \omega = \sqrt{\frac{5a}{m}}$$

and at $x = -4a$,

$$V''(4a) = 11a \implies \omega = \sqrt{\frac{11a}{m}}.$$

Solution 5:

(a) It is easy to see that $y = L\cos\theta$ and $x = L\sin\theta$. For equation of motion, we know that torque on the mass is only due to gravity which is $-Mgx$ and $\tau = I\alpha = mL^2\ddot{\theta}$. Therefore we have

$$mL^2\ddot{\theta} = -mgL\sin\theta \implies \ddot{\theta} + \frac{g\sin\theta}{L} = 0$$

(b) For small angles $\sin\theta \approx \theta$, therefore simple harmonic equation of motion becomes

$$\ddot{\theta} + \frac{g}{L}\theta = 0$$

(c) Taylor series expansion for $\sin\theta$ is

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - ...$$

For $\frac{\theta^3}{3!} \ll \theta$ we have $\sin\theta \approx \theta$ From the simple harmonic equation of motion we can see that

$$\omega = \sqrt{\frac{g}{L}} \implies T = 2\pi\sqrt{\frac{L}{g}}$$

(d)

(e) Let us say that x is the horizontal distance from the pendulum to the central vertical axis. Then, we can write newton's first law as

$$F = m\ddot{x}.$$

F will just be the net horizontal force which can be found by balancing the forces. The net vertical force is 0, so

$$T\cos\theta = mg \implies T = \frac{mg}{\cos\theta}$$

where θ is the angle between the central vertical axis and the rope. The net horizontal force is

$$-T\sin\theta = \frac{-mg\sin\theta}{\cos\theta} = -mg\tan\theta.$$

θ is $\tan^{-1}(\frac{x}{\sqrt{L^2-x^2}})$, so the equation of motion is

$$-mg\tan\theta = m\ddot{x} \implies \boxed{\frac{-mgx}{\sqrt{L^2-x^2}} = m\ddot{x}}$$

(f) If x is small, then we can approximate our equation of motion by getting rid of any terms with x to a high power, so we can approximate our equation of motion to

$$\frac{-mgx}{L} = m\ddot{x} \implies \frac{gx}{L} + \ddot{x} = 0.$$

Problem Set 2

Solution 1:

(a) Note that writing Kirchhoff's Voltage Law in the clockwise direction tells us that $\sum V = 0$ which means

$$\frac{Q(t)}{C} - L\frac{dI}{dt} - IR = 0 \implies \boxed{L\ddot{Q}(t) + R\dot{Q}(t) + \frac{1}{C}Q = 0}.$$

Here we have used the fact that $-\dot{Q} = I$.

(b) The oscillations are probably under-damped as the resistance will be small as compared to the effects of voltage. If it is anything else, there won't be any oscillations in the RLC circuit which is unlikely to happen.

(c) This form represents the general equation of oscillations of θ with a drag force of $-b\dot{\theta}(t)$ which is written as

$$\ddot{\theta} + \Gamma\dot{\theta} + \omega_0^2\theta = 0.$$

Here, we see that

$$\Gamma = \frac{R}{L},$$
$$\omega_0^2 = \frac{1}{LC} \implies \omega_0 = \frac{1}{\sqrt{LC}}$$

We note that

$$Q(t) = \text{Re}[Z(t)], \qquad Z(t) = e^{i\alpha t}$$

where the solution for α is given as

$$\alpha = \frac{i\Gamma}{2} \pm \sqrt{\omega_0^2 - \frac{\Gamma^2}{4}}.$$

Since the oscillations are under-damped or $\omega_0^2 > \Gamma^2/4$, we have that

$$Q(t) = (A\cos(\omega t + \phi)) \cdot e^{-\frac{\Gamma}{2}t} = \boxed{\left(A\cos\left(\frac{1}{\sqrt{LC}}t + \phi\right)\right) \cdot e^{-\frac{R}{2L}t}}$$

where A and ϕ are determined from initial conditions. For the problem given to us $Q(0) = CV_c(t = 0) = 10^{-5}\text{C}$ and $\dot{Q}(0) = -0.5\text{A}$, therefore we have $Q(0) = A\cos(\phi)$ and $\dot{Q}(0) = -A\sqrt{\frac{1}{LC}}\sin(\phi) - A\frac{R}{2L}\cos(\phi)$

Solution 2:

(a) When the crystal is displaced a distance x from the equilibrium position, our equation of motion will be predetermined by

$$M\ddot{x} = F - k_{\text{eff}}x - Mg.$$

Substituting $F = MA_0\cos(\omega_d t)$ and $k_{\text{eff}} = 4k$ into our equation gives us our equation of motion to be

$$\boxed{M\ddot{x} = MA_0\cos(\omega_d t) - 4k(A_0 - x) - Mg}.$$

(b) Now, since the homogeneous solution dies out eventually there should be a very small drag force, we add the drag force factor of $-\gamma\dot{x}$ to our equation. We now have our equation of motion to be

$$M\ddot{x} = MA_0\cos(\omega_d t) - 4kx - Mg - \gamma\dot{x}$$

and rearranging gives us

$$M\ddot{x} + \gamma\dot{x} + 4kx = MA_0\cos(\omega_d t).$$

Note that this equation is very similar to the one discussed and solved in lecture 3:

$$\ddot{\theta} + \Gamma\dot{\theta} + \omega_0^2\theta = f_0\cos(\omega_d t).$$

Dividing by M on both sides of our obtained equation gives us the form

$$\ddot{x} + \frac{\gamma}{M}\dot{x} + \frac{4k}{M}x = A_0\cos(\omega_d t)$$

which now tells us that

$$\Gamma = \frac{\gamma}{M},$$
$$\omega_0^2 = \frac{4k}{M} \implies \omega_0 = \sqrt{\frac{4k}{M}}$$
$$f_0 = A_0.$$

As solved in the lecture, the amplitude of oscillations of this equation is given by

$$A = \frac{f_0}{\sqrt{(\omega_0^2 - \omega_d^2)^2 + \omega_d^2\Gamma^2}}.$$

Therefore, by substituting, we find that

$$A = \frac{A_0}{\sqrt{(4k/M - \omega_d^2)^2 + \gamma^2\omega_d^2/M}}$$

As γ is very small

$$\boxed{A \approx \frac{A_0}{4k/M - \omega_d^2}}$$

(c) To decrease the amplitude we have to increase k which is done by reducing length of spring

$$A_f = \frac{A_i}{10} \implies 10\left(\frac{k}{M} - \omega_d^2\right) = \frac{k_f}{M} - \omega_d^2 \implies k_f = 10k - 9M\omega_d^2$$

We know $k \propto \frac{1}{L}$ therefore

$$\frac{L_f}{L_i} = \frac{k}{10k - 9M\omega_d^2} \approx \frac{1}{10}\left(1 + \frac{9M\omega_d^2}{10k}\right)$$

(d) Like we did in part b,

$$A_f = \frac{A_0}{\sqrt{(4k/M - \omega_d^2)^2 + b^2\omega_d^2/M}} = \frac{A_i}{10}$$

Here $A_i = \frac{A_0}{4K/M - \omega_d^2}$ Therefore

$$99\left(\frac{4K}{M} - \omega_d^2\right) = b^2\frac{\omega_d^2}{M} \implies b = \sqrt{\frac{M}{99\omega_d^2\left(\frac{4K}{M} - \omega_d^2\right)^2}}$$

For $\frac{4K}{M} \gg \omega_d^2$

$$b \approx \sqrt{\frac{M^2}{396K\omega_d^2}\left(1 + \frac{M\omega_d^2}{4K}\right)}$$

Solution 3:

(a) Note that the traffic signal exhibits under damped oscillations which means the amplitude follows the equation

$$x(t) = Ae^{-\Gamma t/2}\cos(\omega t + \phi).$$

After 4 seconds, the amplitude $A \to e^{-1}A$ which means

$$e^{-1}A = Ae^{-2\Gamma}\cos(4\omega + \phi).$$

Since there is no phase difference, $\phi = 0$. Therefore,

$$e^{2\Gamma - 1} = \cos(4\omega) \implies \omega = \frac{1}{4}\arccos\left(e^{2\Gamma-1}\right).$$

Solution 4:

(a) Let the mass-spring system be initially at rest at the beginning. After a time t, the right end (x_{end}) will move a leftward distance of $d(t) = \Delta \sin \omega t$. Let us place the origin at the equilibrium position of the mass. At time t, the right spring will be compressed by an effective length $x = x_0 - \Delta \sin \omega t$ while the left spring remains a rest length of x_0. The drag force works against these two spring forces that are directed in the left direction, so the drag force moves in the right direction with a force of $b\dot{x}$. We now write the equation of motion in both the $\hat{x}$ and $\hat{y}$ direction. In the $\hat{x}$ direction, we have

$$m\ddot{x} = b\dot{x} - (kx_0 + k(x_0 - \Delta \sin \omega t)).$$

This gives us our equation of motion to be

$$\boxed{m\ddot{x} - b\dot{x} + 2kx_0 = k\Delta \sin \omega t}.$$

In the $\hat{y}$ direction, there will be a vertical downward force of $\vec{F}_g$ and a normal force that counteracts this $\vec{F}_N$. We then have

$$m\ddot{y} = \vec{F}_g + \vec{F}_N = -mg\hat{y} + mg\hat{y} = 0.$$

(b) We can guess a solution of the form

$$x(t) = x_{\text{free}} + x_{\text{driven}} = A\cos(\omega_d t - \delta) + B\sin(\omega_d t + \alpha).$$

We see that A and B and the phase constants δ and α all depend on the oscillator properties while ω_d depends on the properties of the external force.

Solution 5:

(a) Let x_1 be the vertical displacement of the mass m_1 and let x_2 be the vertical displacement of the mass m_2. We then can write the equation of motion as

$$\begin{aligned} m\ddot{x}_1 &= k_C x_1 + k_B(x_1 - x_2) \\ m\ddot{x}_2 &= k_A x_2 \end{aligned}$$

We can rewrite this to isolate the equations for both x_1 and x_2.

$$\begin{aligned} m\ddot{x}_1 &= (k_C + k_B)x_1 - k_B x_2 \\ m\ddot{x}_2 &= 0 + k_A x_2 \end{aligned}$$

(b) Now, let us write this in matrix format. The M matrix is denoted by

$$M = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}.$$

From our equations of motion, the K matrix has the form

$$K = \begin{pmatrix} -k_B - k_C & k_B \\ 0 & -k_A \end{pmatrix}.$$

Therefore,

$$M^{-1}K = \begin{pmatrix} -k_B/m - k_C/m & k_B/m \\ 0 & -k_A/m \end{pmatrix}$$

This must equal to $\omega^2 I$, or in other words,

$$\begin{pmatrix} -k_B/m - k_C/m & k_B/m \\ 0 & -k_A/m \end{pmatrix} = \omega^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

(c) The matrix $M^{-1}K - \omega^2 I$ is

$$M^{-1}K - \omega^2 I = \begin{pmatrix} -k_B/m - k_C/m - \omega^2 & k_B/m \\ 0 & -k_A/m - \omega^2 \end{pmatrix}.$$

To find the eigenvalues of M^1K, we form the determinant

$$\begin{aligned} \det\left[M^{-1}K - \omega^2 I\right] &= \begin{vmatrix} -k_B/m - k_C/m - \omega^2 & k_B/m \\ 0 & -k_A/m - \omega^2 \end{vmatrix} \\ &= (-k_B/m - k_C/m - \omega^2)(-k_A/m - \omega^2) - (k_B/m)(0) \\ &= (-k_B/m - k_C/m - \omega^2)(-k_A/m - \omega^2) = 0. \end{aligned}$$

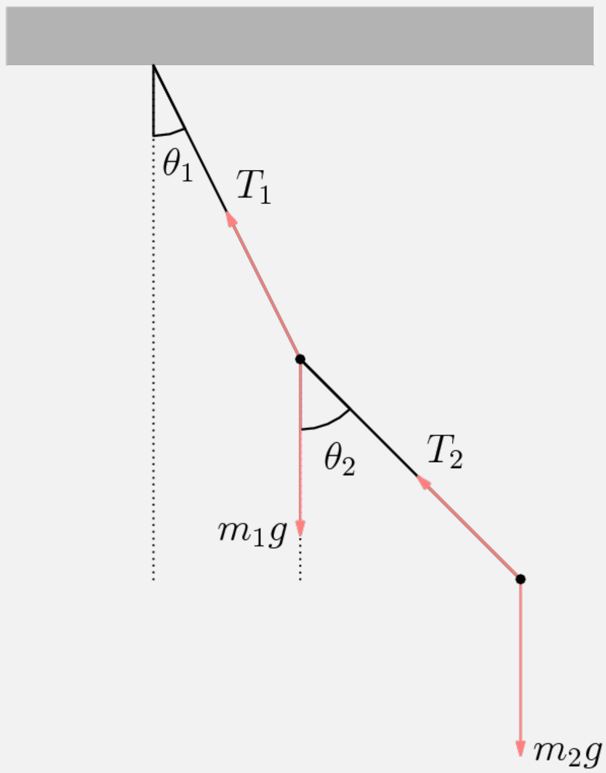
Thus, the angular frequencies of the normal modes are

$$\omega_1^2 = \frac{k_B}{m} + \frac{k_C}{m}, \quad \omega_2^2 = \frac{k_A}{m}.$$

Problem Set 3

Solution 1:

(a) We begin by first drawing a diagram:



Since the angles are small, it holds that

$$\sin\theta \approx \theta, \quad \cos\theta \approx 1.$$

Let us also define

$$\sin\theta_1 = \frac{X_1}{L}, \quad \sin\theta_2 = \frac{X_2 - X_1}{L}.$$

The angles are small so $T_1 \approx (M_1 + M_2)g$ (as it holds both masses) and $T_2 \approx M_2 g$ (as it holds only the mass M_2). We can now write our equations of motion. We can start with the lower mass. The only force that attempts to bring the second mass back to equilibrium is the horizontal component of T_2. Therefore, we have

$$M_2\ddot{X}_2 = -T_2\sin\theta_2 = -\frac{M_2 g}{\ell}(X_2 - X_1).$$

If we define $\omega^2 \equiv g/L$, we find

$$\ddot{X}_2 + \omega_0^2 X_2 - \omega_0^2 X_1 = 0.$$

For the first mass, we have two forces. Although not labeled in the diagram there is a component T_2 that is directed along the rod and away from the first mass. Therefore, the two forces T_1 and T_2 fight for equilibrium and our equation of motion can be given as

$$M_1\ddot{X}_1 = -T_1\sin\theta_1 + T_2\sin\theta_2$$
$$M_1\ddot{X}_1 = -2(M_1 + M_2)g\frac{X_1}{L} + M_2 g\frac{X_2 - X_1}{L}$$
$$0 = \ddot{X}_1 + \frac{M_1 + 2M_2}{M_1}X_1\omega_0^2 - \frac{M_2}{M_1}X_2\omega_0^2$$

(b) Using the definition of a normal mode

$$\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \mathrm{Re}\left[\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} e^{i(\omega t+\phi)}\right]$$

we can rearrange to find that

$$0 = \left(\frac{M_1 + 2M_2}{M_1}\omega_0^2 - \omega^2\right) A_1 - \frac{M_2}{M_1}\omega_0^2 A_2$$
$$0 = -\omega_0^2 A_1 + \left(\omega_0^2 - \omega^2\right)$$

Now, rewrite in matrix format

$$\begin{pmatrix} \frac{M_1+2M_2}{M_1}\omega_0^2 - \omega^2 & -\frac{M_2}{M_1}\omega_0^2 \\ -\omega_0^2 & \omega_0^2 - \omega^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0.$$

To get a solution, we need to solve the equation where the determinant of the left matrix is zero.

$$\begin{vmatrix} \frac{M_1+2M_2}{M_1}\omega_0^2 - \omega^2 & -\frac{M_2}{M_1}\omega_0^2 \\ -\omega_0^2 & \omega_0^2 - \omega^2 \end{vmatrix} = 0.$$

$$\left(\frac{M_1 + 2M_2}{M_1}\omega_0^2 - \omega^2\right)\left(\omega_0^2 - \omega^2\right) - \frac{M_2}{M_1}\omega_0^4 = 0$$

Rearranging yields the equation

$$\omega^4 - \frac{2(M_1 + M_2)}{M_1}\omega_0^2\omega^2 + \frac{M_1 + M_2}{M_1}\omega_0^4 = 0.$$

Using substitution tells us that the roots of this equation are

$$\omega^2 = \frac{(M_2 + M_1) \pm \sqrt{M_2^2 + M_1 M_2}}{M1}\omega_0^2 \implies \omega^2 = (1+\alpha) \pm \sqrt{1+\alpha^2}$$

Here we used $\alpha = \frac{M_2}{M_1}$

Solution 2:

(a) To find the normal modes of frequencies, we must first find the equations of motions. Note that we can use symmetry arguments after we find the equation of motion in one circuit. The flux in the first circuit is given by the sum of the self and mutual inductance. Mutual inductance means that the two inductors share flux or more specifically, the relationship in flux between an inductor L_1 and the current I_2 is given by

$$\Phi_{m_1} = MI_2.$$

The total flux in circuit 1 is then given by

$$\Phi_{\mathrm{tot}_1} = \Phi_{s_1} + \Phi_{m_1} = LI_1 + MI_2.$$

Taking a time derivative of this equation with using $\dot{Q} = -I$ results in

$$\dot{\Phi}_{\mathrm{tot}_1} = -L\ddot{Q}_1 - M\ddot{Q}_2.$$

Using the fact that $\dot{\Phi} = V$ and $V = Q/C$, we result in the equation

$$\frac{Q_1}{C} = -L\ddot{Q}_1 - M\ddot{Q}_2.$$

Using $\omega \equiv 1/\sqrt{LC}$, we can get the equations (by symmetry)

$$\omega^2 Q_1 = -\ddot{Q}_1 - \frac{M}{L}\ddot{Q}_2$$
$$\omega^2 Q_2 = -\ddot{Q}_2 - \frac{M}{L}\ddot{Q}_1$$

Adding these two equations gives

$$-\omega^2(Q_1 + Q_2) = \left(1 + \frac{M}{L}\right)(\ddot{Q}_1 + \ddot{Q}_2$$

Therefore one normal mode is $\omega_+^2 = \frac{L\omega^2}{L+M} \implies \omega_+ = \sqrt{\frac{1}{C(L+M)}}$

Subtracting the two equations gives

$$-\omega^2(Q_1 - Q_2) = \left(1 - \frac{M}{L}\right)(\ddot{Q}_1 + \ddot{Q}_2)$$

Therefore second normal mode is $\omega_- = \sqrt{\frac{L\omega^2}{L-M}} \implies \omega_- = \sqrt{\frac{1}{C(L-M)}}$

(b)

Solution 3:

(a) Let the distances of each of the masses from their equilibrium points be denoted by x_1, x_2, and x_3 in the clockwise direction. We then see that the equations of motions of the masses are defined by

$$\begin{aligned} m\ddot{x}_1 &= -k(x_1 - x_2) - k(x_1 - x_3) \\ m\ddot{x}_2 &= -k(x_2 - x_1) - k(x_2 - x_3) \\ m\ddot{x}_3 &= -k(x_3 - x_1) - k(x_3 - x_2) \end{aligned}$$

(b) First, we rewrite our equation of motion to isolate x_1, x_2, and x_3. By rewriting, we have

$$\begin{aligned} m\ddot{x}_1 &= -2kx_1 + kx_2 + kx_3 \\ m\ddot{x}_2 &= kx_1 - 2kx_2 + kx_3 \\ m\ddot{x}_3 &= kx_1 + kx_2 - 2kx_3 \end{aligned}$$

Using the definition of a normal mode

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathrm{Re}\left[\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} e^{i(\omega t + \phi)}\right]$$

we can then rearrange to find

$$\begin{aligned} -m\omega^2 A_1 &= -2kA_1 + kA_2 + kA_3 & & 0 = \left(-2k + 2m\omega^2\right) A_1 + kA_2 + kA_3 \\ -m\omega^2 A_2 &= kA_1 - 2kA_2 + kA_3 & \Rightarrow \quad & 0 = -kA_1 + (-2k + m\omega^2)A_2 + kA_3 \\ -m\omega^2 A_3 &= kA_1 + kA_2 - 2kA_3 & & 0 = kA_1 + kA_2 + (-2k + m\omega^2)A_3 \end{aligned}$$

Now, rewrite in matrix format

$$\begin{pmatrix} -2k + 2m\omega^2 & k & k \\ -k & -2k + m\omega^2 & k \\ k & k & -2k + m\omega^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = 0$$

To get a solution we need to solve the equation where the determinant of the left matrix is zero.

$$\begin{vmatrix} -2k + 2m\omega^2 & k & k \\ k & -2k + m\omega^2 & k \\ k & k & -2k + m\omega^2 \end{vmatrix} = 0$$

If we define $\omega_0^2 = \frac{k}{m}$, then after dividing the matrix by m, we yield the equation

$$\begin{vmatrix} -2\omega_0^2 + 2\omega^2 & \omega_0^2 & \omega_0^2 \\ \omega_0^2 & -2\omega_0^2 + \omega^2 & \omega_0^2 \\ \omega_0^2 & \omega_0^2 & -2\omega_0^2 + \omega^2 \end{vmatrix} = 0.$$

After evaluating the determinant we result in the equation

$$(-2\omega_0^2 + 2\omega^4)[(-2\omega_0^2 + \omega^2)^2 - \omega_0^4] = 0.$$

We now gain two roots of $\omega^2 = 0$ and $\omega^2 = 3\omega_0^2$.

(c) We see that in the $\omega^2 = 0$ root, the normal mode is defined by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} (At + B)$$

and corresponds to the masses sliding around the circle at a constant speed. Therefore, there are not really any oscillations exhibited in this mode.

Solution 4:

(a) Let us first find the equations of motion without the driving force and then add it in when we are done. Let us denote the upper mass by m_1 and the lower mass by m_2. We then see that the forces produced by the springs are $ky_1(t)$ and $ky(2)$. The forces in the spring are then $ky_1(t)$ and $k(y_2(t) - y_1(t))$ respectively. The gravitational force only creates a shift in the equilibrium position but this doesn't affect our equation of motion. This then tells us that our coupled differential equations are:

$$\begin{aligned} m_1\ddot{y}_1 &= +k(y_2(t) - y_1(t)) - ky_1(t) + F_0 \cos(\omega_d t) \\ m_2\ddot{y}_2 &= -k(y_2(t) - y_1(t)) \end{aligned}$$

Defining $m_1 = m_2 = m$ and $\omega_0^2 \equiv k/m$, we have

$$\begin{aligned} \ddot{y}_1 &= +\omega_0^2(y_2(t) - 2y_1(t)) + \frac{F_0}{m}\cos(\omega_d t) \\ \ddot{y}_2 &= -\omega_0^2(y_2(t) - y_1(t)) \end{aligned}$$

(b) We define

$$M = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}, \quad K = \begin{pmatrix} +2k & -k \\ -k & +k \end{pmatrix}, \quad Y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

With a driving force, our matrix equation is given by

$$M\ddot{Y} = -KY + F\cos(\omega_d t).$$

Let us try to consider solutions for ω without a driving force. Using the definition of a normal mode

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathrm{Re}\left[\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} e^{i(\omega t + \phi)}\right]$$

we can get the equation down to

$$\begin{aligned} -\omega^2 A_1 &= -2A_1\omega_0^2 + A_2\omega_0^2 & \qquad 0 &= \left(-2\omega_0^2 + \omega^2\right) A_1 + \omega_0^2 A_2 \\ -\omega^2 A_2 &= A_1\omega_0^2 - A_2\omega_0^2 & \Rightarrow \quad 0 &= \omega_0^2 A_1 + (\omega^2 - \omega_0^2) A_2 \end{aligned}$$

Normalizing this into a matrix equation tell us

$$\begin{pmatrix} -2\omega_0^2 + \omega^2 & \omega_0^2 \\ \omega_0^2 & -\omega_0^2 + \omega^2 \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = 0.$$

To get a solution we need to solve the equation where the determinant of the left matrix is zero

$$\begin{vmatrix} -2\omega_0^2 + \omega^2 & \omega_0^2 \\ \omega_0^2 & -\omega_0^2 + \omega^2 \end{vmatrix} = 0.$$

Reducing this, we get

$$(-\omega_0^2 + \omega^2)(-2\omega_0^2 + \omega^2) - \omega_0^4 = -.$$

Simplifying gives us

$$\begin{aligned} 2\omega_0^4 - 2\omega^2\omega_0^2 - \omega_0^2\omega^2 + \omega^4 - \omega_0^4 &= 0 \\ \omega_0^4 - 3\omega^2\omega_0^2 + \omega^4 &= 0 \end{aligned}$$

This now implies that the roots of this equation are now

$$\omega_+ = \sqrt{\frac{(3+\sqrt{5})k}{2m}}, \quad \omega_- = \sqrt{\frac{(3-\sqrt{5})k}{2m}}.$$

Using these solutions and the fact that

$$\det\left(M^{-1}K - \omega^2 I\right) = (\omega^2 - \omega_+^2)(\omega^2 - \omega_-^2) = 0$$

to further simplify this problem. Note now that

$$M^{-1}K = \begin{pmatrix} 2\omega_0^2 & -\omega_0^2 \\ -\omega_0^2 & \omega_0^2 \end{pmatrix}, \quad M^{-1}F = \begin{pmatrix} F_0/m \\ 0 \end{pmatrix}.$$

Using

$$(M^{-1}K - \omega^2 I)B = M^{-1}F$$

to then expand into matrix form gives us

$$\begin{pmatrix} 2\omega_0^2 - \omega_d^2 & -\omega_0^2 \\ -\omega_0^2 & \omega_0^2 - \omega_d^2 \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} = \begin{pmatrix} F_0/m \\ 0 \end{pmatrix}$$

We can go ahead and solve it directly to get B_1 and B_2 or we can use "Cramer's Rule" which is a useful rule when solving a large number of coupled oscillators. First define:

$$E^{\leftrightarrow} = \begin{pmatrix} 2\omega_0^2 - \omega_d^2 & -\omega_0^2 \\ -\omega_0^2 & \omega_0^2 - \omega_d^2 \end{pmatrix}, \quad D^{\rightarrow} = \begin{pmatrix} F_0/m \\ 0 \end{pmatrix}.$$

To use Cramer's rule, use one column from $E^{\leftrightarrow}$ and $D^{\rightarrow}$

$$\begin{aligned} B_1 &= \frac{|(D^{\rightarrow})()|}{\det E^{\leftrightarrow}} \\ &= \frac{\begin{pmatrix} F_0/m & -\omega_0^2 \\ 0 & \omega_0^2 - \omega_d^2 \end{pmatrix}}{(\omega^2 - \omega_+^2)(\omega^2 - \omega_-^2)} \\ &= \frac{\frac{F_0}{m}(\omega_0^2 - \omega_d^2)}{(\omega^2 - \omega_+^2)(\omega^2 - \omega_-^2)} \end{aligned}$$

Which explodes when $\omega_d = \omega_+, \omega_-$ which are the frequencies of the normal modes. Similarly:

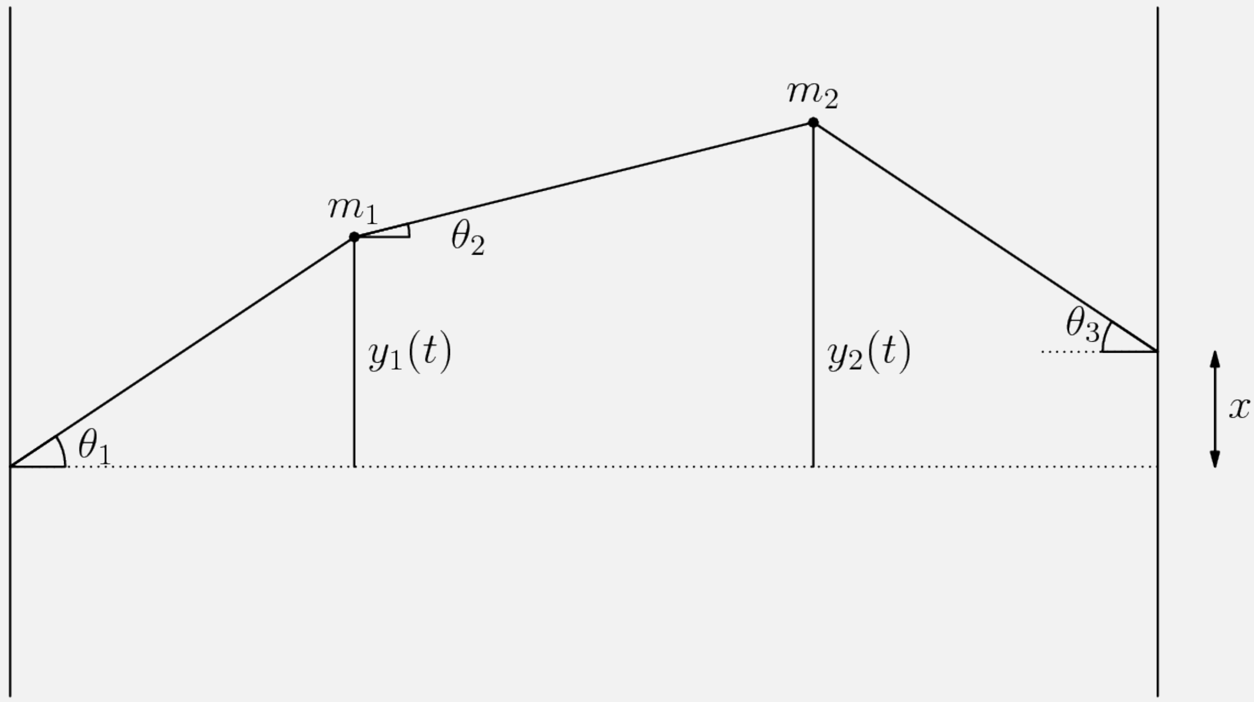
$$\begin{aligned} B_2 &= \frac{|()(D^{\rightarrow})|}{\det E^{\leftrightarrow}} \\ &= \frac{\begin{pmatrix} 2\omega_0^2 & F_0/m \\ -\omega_0^2 & 0 \end{pmatrix}}{(\omega^2 - \omega_+^2)(\omega^2 - \omega_-^2)} \\ &= \frac{\frac{F_0}{m}\left(\frac{\omega_0^2}{m}\right)}{(\omega^2 - \omega_+^2)(\omega^2 - \omega_-^2)} \end{aligned}$$

The full solution is given by

$$y_1 = \alpha\cos(\omega_+ t + \phi) + \beta\cos(\omega_- t + \phi) + B_1\cos(\omega_d t)$$

Solution 5:

(a) Let us first draw a diagram.



Since the angles are small, and the horizontal components of tension are approximately the same and can be approximated by

$$T_1 \cos\theta_1 \approx T_2 \cos\theta_2 \approx T_3 \cos\theta_3 \implies T_1 \approx T_2 \approx T_3 = T.$$

We can approximate the sines of the angles by

$$\sin\theta_1 = \frac{y_1(t)}{a}$$
$$\sin\theta_2 = \frac{y_2(t) - y_1(t)}{a}$$
$$\sin\theta_3 = \frac{y_2(t)}{a}$$

where a is the length of each component of the rope. For m_1, we have

$$m_1\ddot{y}_1 = T_2 \sin\theta_2 - T_1 \sin\theta_1$$

and using our approximations tells us that

$$\boxed{\ddot{y}_1 = \frac{T}{m_1 a}(y_2 - 2y_1)}.$$

Similarly, we have for m_2,

$$\boxed{\ddot{y}_2 = \frac{T}{m_2 a}(y_1 - 2y_2)}.$$

Problem Set 4

Solution 1:

(a) There are two boundary conditions.

Solution 2:

(a) Since the matrices commute, the modes will be simultaneous eigenvectors of the symmetry transformation and the interaction. There are then two modes. These two solution are $x(t)$ and $\tilde{x}(t) = \mathcal{S}x(t)$.

(b) When we have a symmetry matrix either we don't undergo any transformation on the eigenvalue $\lambda = 1$, or it will undergo a transformation on the eigenvalue $\lambda = -1$. [a] Since the 1s of the symmetry matrix are off the diagonal, when we multiply $\mathcal{S}$ by the components of the eigenvector we will switch the eigenvalues. For example:

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} -B \\ A \end{pmatrix}$$

We need the absolute value of A and B to be the same if we want it to only change by a factor of 1 or -1 upon reflection. Therefore, if we try something such as

$$\begin{pmatrix} A \\ A \end{pmatrix}$$

and multiply by the symmetry matrix, we get

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} A \\ A \end{pmatrix} = \begin{pmatrix} -A \\ -A \end{pmatrix}$$

The ratio of the amplitudes of the components of the eigenvector are then 1.

[a] This is justified because $\det(A - \lambda I) = \lambda^2 - 1 = 0 \implies \lambda = 1, \lambda = -1$.

Solution 3:

(a) We have two boundary conditions:

- At $x = 0$, $\frac{\partial \psi(0,t)}{\partial x} = 0$.
- At $x = L$, $\frac{\partial \psi(L,t)}{\partial x} = 0$.

By symmetry, the wavefunction is given by

$$\psi_m(x,t) = A_m \sin(\omega_m t + \beta_m) \sin(k_m x + \alpha_m).$$

Taking the partial derivative of this with respect to x gives us

$$\frac{\partial \psi_m}{\partial x} = A_m k_m \sin(\omega_m t + \beta_m) \cos(k_m x + \alpha_m).$$

We now can apply the boundary conditions.

- At $x = 0$:
$$\frac{\partial \psi_m(0,t)}{\partial x} = A_m k_m \sin(\omega_m t + \beta_m) \cos(\alpha_m) = 0.$$
Therefore, we require that
$$\cos(\alpha_m) = 0 \implies \alpha_m = \pi m - \frac{\pi}{2}.$$

- At $x = L$:

$$\frac{\partial \psi_m}{\partial x} = A_m k_m \sin(\omega_m t + \beta_m) \cos(k_m L + \pi m - \pi/2) = 0.$$

We require that

$$k_m L + \pi m - \frac{\pi}{2} = \pi n - \frac{\pi}{2}.$$

This means that

$$k_m = \frac{\pi(n-m)}{L}, \quad a = n - m = 1, 2, 3, \ldots$$

or

$$k_m = \frac{\pi a}{L}.$$

(b) We can draw each mode.

- For the first mode $a = 1$, there w

Solution 4:

(a)

Problem Set 5

Solution 1:

(a)

Solution 2:

(a)

Solution 3:

(a)

Solution 4:

(a)

Solution 5:

(a) Let us analyze the forces involved. There is a vertical drag force $b\dot{y}$, a tension directed downwards at an angle θ to the horizontal, and a horizontal normal force that acts on the pole in the opposite direction of the tension.

Applying Newton's law in the vertical direction gives us

$$m\ddot{y} = -T\sin\theta + F_d.$$

Writing the force in differential form and assuming small angles yields,

$$m\ddot{y} = -T\frac{\partial y}{\partial x} - b\frac{\partial y}{\partial t}.$$

The mass of the ring is negligible and thus

$$-T\frac{\partial y}{\partial x} - b\frac{\partial y}{\partial t} = 0 \implies \boxed{\frac{\partial y}{\partial x} = -\frac{b}{T}\frac{\partial y}{\partial t}}.$$

(b) The wave can be written as a superposition of the incident and reflected pulse. In other words,

$$y(x,t) = f(vt-x) + g(vt+x).$$

The reflected wave $g(x+vt)$ is unknown, but we can use boundary conditions used in part (a) to solve for it.

$$\frac{\partial y}{\partial x} = f'(vt-x) + g'(vt+x)$$
$$\frac{\partial y}{\partial t} = v(f'(vt-x) + g'(vt+x))$$

Defining are coordinate system such that the hoop is at $x=0$, we then can then write

$$\frac{\partial y}{\partial x} = -\frac{b}{T}\frac{\partial y}{\partial T} \implies f'(vt) + g'(vt) = -\frac{bv}{T}(f'(vt) + g'(vt)).$$

Isolating $g(vt)$ tells us that

$$g'(vt) = \frac{bv-T}{bv+T}f'(vt).$$

To now solve for g in terms of f we must integrate this expression. Substituting $u = vt$ gives us

$$\int g'(u)du = \int \frac{bv - T}{bv + T} f'(u)du \implies g(vt) = \boxed{\frac{bv - T}{bv + T} f(vt)}.$$

Problem Set 6

Solution 1:

(a)

Solution 2:

(a) The most straightforward way to prove the curl of a curl vector identity

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - (\vec{\nabla} \cdot \vec{\nabla})\vec{A}$$

is by simply expanding out all of the terms. We start off with the most basic identity

$$\vec{\nabla} \cdot \vec{A} = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\vec{A}.$$

Let us denote this by ξ as our terms will start to get messy. When we take $\vec{\nabla}(\vec{\nabla} \cdot \vec{A}) = \vec{\nabla} \cdot \vec{\xi}$, we result in

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{A}) = \nabla \xi = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\vec{\xi}.$$

By expanding outwards, we get

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{A}) = \left[\frac{\partial}{\partial x}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)\right]\hat{x} + \left[\frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)\right]\hat{y} + \left[\frac{\partial}{\partial z}\left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)\right]\hat{z}$$

Simplifying gives us

$$\vec{\nabla}(\vec{\nabla} \cdot \vec{A}) = \left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_y}{\partial x \partial y} + \frac{\partial^2 A_z}{\partial x \partial z}\right)\hat{x} + \left(\frac{\partial^2 A_x}{\partial x \partial y} + \frac{\partial^2 A_y}{\partial^2 y} + \frac{\partial^2 A_z}{\partial y \partial z}\right)\hat{y} + \left(\frac{\partial^2 A_x}{\partial x \partial z} + \frac{\partial^2 A_y}{\partial y \partial z} + \frac{\partial^2 A_z}{\partial z^2}\right)\hat{z}$$

Alternatively, when taking $(\vec{\nabla} \cdot \vec{\nabla})\vec{A}$, we result in the laplacian in cartesian coordinates for each direction.

$$(\vec{\nabla} \cdot \vec{\nabla})\vec{A} = \left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2}\right)\hat{x} + \left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2}\right)\hat{y} + \left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_y}{\partial y^2} + \frac{\partial^2 A_z}{\partial z^2}\right)\hat{z}.$$

We now have the right-hand side of our equation. Now, we need to expand outwards the left-hand side. First, note that

$$\nabla \times \vec{A} = \left(\frac{\partial A_z}{\partial z} - \frac{\partial A_y}{\partial y}\right)\hat{x} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{y} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{z}.$$

From this, we can note that

$$\begin{aligned}
\nabla \times (\nabla \times \vec{A}) = &\left[\frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\right]\hat{x} \\
&+ \left[\frac{\partial}{\partial z}\left(\frac{\partial A_z}{\partial z} - \frac{\partial A_y}{\partial y}\right) - \frac{\partial}{\partial x}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\right]\hat{y} \\
&+ \left[\frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial z} - \frac{\partial A_y}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\right]\hat{z}
\end{aligned}$$

Multiplying across now gives us

$$\begin{aligned}
\nabla \times (\nabla \times \vec{A}) = &\left[\left(\frac{\partial^2 A_y}{\partial x \partial z} - \frac{\partial^2 A_x}{\partial y \partial z}\right) - \left(\frac{\partial^2 A_x}{\partial y \partial z} - \frac{\partial^2 A_z}{\partial x \partial y}\right)\right]\hat{x} \\
&+ \left[\left(\frac{\partial^2 A_z}{\partial^2 z} - \frac{\partial^2 A_y}{\partial y \partial z}\right) - \left(\frac{\partial^2 A_y}{\partial x^2} - \frac{\partial^2 A_x}{\partial x \partial y}\right)\right]\hat{y} \\
&+ \left[\left(\frac{\partial^2 A_z}{\partial x \partial z} - \frac{\partial^2 A_y}{\partial x \partial y}\right) - \left(\frac{\partial^2 A_x}{\partial y \partial z} - \frac{\partial^2 A_z}{\partial x \partial y}\right)\right]\hat{z}
\end{aligned}$$

We now finally conclude that since the terms on both sides are the same, then

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - (\vec{\nabla} \cdot \vec{\nabla})\vec{A}$$

holds true.

(b) In a vacuum, we have

$$\vec{\nabla} \times \overbrace{(\vec{\nabla} \times \vec{B})}^{\mu_0 \epsilon_0 \partial \vec{B}/\partial t} = \vec{\nabla}\overbrace{(\vec{\nabla} \cdot \vec{B})}^{0} - (\vec{\nabla}^2)\vec{B}$$

These replacements are done by the vacuum Maxwell's equations. Now, we are simply left with

$$\vec{\nabla} \times \left(\mu_0 \epsilon_0 \frac{\partial \vec{B}}{\partial t}\right) = \mu_0 \epsilon_0 \left[\frac{\partial}{\partial t}(\vec{\nabla} \times \vec{E})\right].$$

Therefore,

$$-\vec{\nabla}^2 B = -\mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2} \implies \vec{\nabla}^2 B = \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}.$$

Solution 3:

(a) To show that the field satisfies the EM wave equation we first write the EM wave equation as

$$\vec{\nabla}^2 E = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}.$$

We see that $\vec{\nabla}^2 E$ is given by

$$\vec{\nabla}^2 E = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \vec{E}.$$

Since the EM wave only goes in the $\hat{y}$ direction, this means that

$$\vec{\nabla}^2 E = \frac{\partial^2 E_y}{\partial x^2} \hat{y} = E f''(x - ct)\hat{y}.$$

Also,

$$\frac{\partial^2 \vec{E}}{\partial t^2} = c^2 E_0 f''(x - ct)\hat{y}.$$

This now clearly satisfies the EM wave equation.

(b) We note that

$$\vec{\nabla} \cdot \vec{E} = \left(\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_z}{\partial z^2}\right) = 0$$

as long as $E_x = 0$.

(c) Note that

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \implies -\frac{\partial \vec{B}}{\partial t} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & E_y & 0 \end{vmatrix}.$$

This tells us that

$$-\frac{\partial \vec{B}}{\partial t} = \frac{\partial E_y}{\partial x} \hat{z} \implies \vec{B} = \boxed{\frac{E_0}{c} f(x - ct)\hat{z}}.$$

Solution 4:

(a)

Problem Set 7

Solution 1:

(a) Note that

$$\frac{g}{k} \approx \frac{Tk}{\rho} \implies \frac{2\pi}{\lambda_{\text{crit}}} \approx \sqrt{\frac{\rho g}{T}}.$$

This means that

$$\lambda_{\text{crit}} = 2\pi\sqrt{\frac{T}{\rho g}}.$$

(b) Remember our equation:

$$v_p^2 = \left(\frac{g}{k} + \frac{Tk}{\rho}\right)\tanh(kh).$$

In the shallow water limit where $\lambda \gg h$ and $kh \ll 1$, we have that

$$\tanh(kh) \approx kh, \quad \text{and} \quad v_p^2(k) = \frac{\omega^2}{k^2}.$$

Note that $\frac{g}{k} \gg \frac{Tk}{\rho}$ so therefore,

$$v_p^2 \approx gh \implies \omega^2 = ghk^2.$$

The phase velocity will be $v_p = \sqrt{gh}$ while the group velocity will be

$$v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \sqrt{gh}.$$

As v_g is constant, the waves are non-dispersive in this case.

(c) In the limit of deep water waves, $h \gg \lambda$ and $kh \gg 1$. Therefore, we have

$$\tan(kg) \approx 1, \quad \text{and} \quad v_p^2 = \frac{\omega^2}{k^2}.$$

This tells us that

$$\omega^2 = gk + \frac{T}{\rho}k^3$$

where $\omega = 2\pi/\lambda$. For very long wavelengths, the k term dominates and then $\omega^2 \approx gk$ and the phase velocity is

$$v_p = \sqrt{\frac{g}{k}}$$

and the group velocity

$$v_g = \frac{1}{2}\sqrt{\frac{g}{k}}.$$

Therefore, $v_g = v_p/2$ and since the group velocity is not constant, the waves are dispersive in this medium.

(d) The k^3 term from our equation in part (c) dominates for $\lambda \ll \lambda_{\text{crit}}$. Then,

$$\omega^2 \approx \frac{T}{\rho k^3}$$

and

$$v_p = \frac{\omega}{k} = \sqrt{\frac{Tk}{\rho}}$$

and

$$v_g = \frac{\mathrm{d}\omega}{\mathrm{d}k} = \frac{3}{2}\sqrt{\frac{Tk}{\rho}}.$$

Therefore, $v_g = \frac{3}{2}v_p$ and since the group velocity is not constant, capillary waves are dispersive.

Solution 3:

(a) To find the range of frequencies, we use the inverse fourier transform:

$$C(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}t f(t) e^{i\omega t}$$

with

$$f(x,t) = \exp\left[-\frac{1}{2}\left(\frac{x-vt}{\sigma}\right)^2\right].$$

Our integral then looks like

$$C(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}t \exp\left[-\frac{1}{2}\left(\frac{x-vt}{\sigma}\right)^2\right] e^{i\omega t}.$$

We can then arrange the integral to be formatted such that:

$$C(\omega) = \frac{1}{2\pi}\mathrm{d}t \int_{-\infty}^{\infty} \exp\left[-\left(\frac{v^2}{2\sigma^2}\right)t^2 + \left(\frac{\omega v}{\sigma^2} + i\omega\right)t - \frac{x^2}{2\sigma^2}\right].$$

In general, if we have an integral in the form of

$$I = \int \mathrm{d}x e^{-ax^2+bx+c}$$

we solve it such that

$$I = \int \mathrm{d}x e^{-a\left(x-\frac{b}{2a}\right)^2} e^{\frac{b^2}{4a}+c} = \sqrt{\frac{\pi}{a}} e^{\frac{b^2}{4a}+c}.$$

In this example, if we let

$$a = \frac{v^2}{2\sigma^2}, \quad b = \frac{\omega v}{\sigma} + i\omega, \quad c = -\frac{x^2}{2\sigma^2}.$$

Therefore,

$$C(\omega) = \sqrt{\frac{\pi}{v^2/2\sigma^2}} \exp\left[\left(\frac{\omega v}{\sigma^2} + i\omega\right)^2 / \frac{4v^2}{2\sigma^2} - \frac{x^2}{2\sigma^2}\right].$$